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Extending Representations of Dense Subalgebras of $C^\star$-Algebras, and Spectral Invariance

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Abstract

We show that if certain topologically irreducible representations of a dense m-convex Fréchet subalgebra A of a $C^\star$-algebra B are contained in $\star$-representations of B on a Hilbert space, then the spectrum of every element of A is the same in either A or B. When B is the $C^\star$-algebra associated with a dynamical system consisting of $\mathbb{Z}^2$ acting on $\mathbb{R}^2$ by linear translations, we show that such representations extend if and only if B is CCR.

Contents

1 Introduction

In [duC1, 1989], it was shown that the algebra of smooth compact operators $\mathcal{K}^\infty$ has a “differentiable” irreducible representation theory similar to the irreducible $\star$-representation theory of the usual $C^\star$-algebra of compact operators. In this paper, E is a *differentiable* Fréchet space representation of

the (Fréchet algebra) A if E is a continuous, nondegenerate A-module such that the canonical map $A\widehat{\otimes}E \to E$ is onto, where $\widehat{\otimes}$ denotes the projective completion of Fréchet spaces. We consider these same types of representations, and a variation on them (see Question C in §2). Question D posed in §5 asks: When is every differentiable topologically irreducible representation of A contained in a $\star$-representation of the $C^\star$-algebra B on a Hilbert space? (For $A = \mathcal{K}^\infty = \mathcal{S}(\mathbb{N}^2)$, the compact operators on $l^2(\mathbb{N})$ with Schwartz kernels (and matrix multiplication), the only such representation is $\mathcal{S}(\mathbb{N})$, where the action is $(\varphi\xi)(n) = \sum_m \varphi(n,m)\xi(m)$. Clearly $\mathcal{S}(\mathbb{N})$ is contained in $l^2(\mathbb{N})$, on which $\mathcal{K}$ is $\star$-represented.) If the answer to Question C or D is "yes", we show in §3 that if A is an m-convex Fréchet algebra, then A must be spectral invariant in B. That is, for $a \in A$, the spectrum $spec(a)$ is the same in either A or B.

In §4, we show that representations *do not* extend for the two familiar examples of the irrational rotation algebra, and the crossed product of $\mathbb{Z}$ acting on the one point compactification of $\mathbb{Z}$ by translation. In the second example, the $C^\star$-crossed product is GCR but not CCR.

A positive answer to the extension problem seems to depend on the $C^\star$-algebra B being CCR. We prove this (in §5) for a simple class of smooth

crossed products. Namely, we consider $\mathbb{Z}^2$ acting on $\mathbb{R}^2$ viz $\alpha_{\vec{n}}(\vec{r}) = \vec{r} + n_1\vec{v}_1 + n_2\vec{v}_2$, where $\vec{v}_1$, $\vec{v}_2$ are vectors in $\mathbb{R}^2$. The Schwartz functions $A = \mathcal{S}(\mathbb{Z}^2, \mathbb{R}^2) = \mathcal{S}(\mathbb{Z}^2, \mathcal{S}(\mathbb{R}^2))$ form a dense m-convex Fréchet subalgebra of the $C^\star$-dynamical system $B = C^\star(\mathbb{Z}^2, \mathbb{R}^2)$, with the natural convolution multiplication. In Theorem 5.3, we show that the differentiable irreducible representations of A extend if and only if B is CCR (which is true if and only if $\vec{v}_1$ is not an irrational multiple of $\vec{v}_2$, and if and only if there are no non-closed $\mathbb{Z}^2$-orbits).

In §6, we give an example when the extension Question C has a positive answer but Question D does not.

2 Posing the extension question

Let B be a $C^\star$-algebra and let A be a dense subalgebra of B. We assume that both algebras are either unital with the same unit, or both non-unital. In the non-unital case, we let $\tilde{A}$, $\tilde{B}$ be the respective unitizations. Otherwise set $\tilde{A} = A$, $\tilde{B} = B$. (Note that if A already has a unit, then by density B is automatically unital with the same unit.) We begin with some standard examples.

Example 2.1. $B = C(M)$, the continuous functions on a compact manifold

M without boundary, with pointwise multiplication, and $A = C^\infty(M)$. A is dense by the Stone-Weierstrass Theorem.

Example 2.2. $B = \mathcal{K}(l^2(\mathbb{N}))$, the compact operators on a separable Hilbert space, and $A = \mathcal{K}^\infty = \{[a_{nm}] \mid \ \|[a]\|_{p,q} = \sum_{n,m} |a_{nm}| n^p m^q < \infty, \quad p, q \in \mathbb{N}\}$. Here the algebra structure is matrix multiplication, and A is dense because it contains all rank one operators $\xi \otimes \eta$, $\xi, \eta \in \mathcal{S}(\mathbb{N})$, where $\mathcal{S}(\mathbb{N}) = \{\phi \colon \mathbb{N} \to \mathbb{C} \mid \|\phi\|_p = \sum_n |\phi(n)| n^p < \infty, \quad p \in \mathbb{N}\}$ denotes the set of Schwartz functions on $\mathbb{N}$.

In both examples, A is actually a $\star$-subalgebra of B, though this will not be assumed in general. It is natural to ask what properties A has in common with B. We look at the representation theory of A, and begin by asking the general question:

Question A. When is every representation of A contained in a $\star$-representation B on a Hilbert space?

First note that the answer to this question is not always "yes". Assume $a \in \tilde{A}$, $a^{-1} \in \tilde{B}$, $a^{-1} \notin \tilde{A}$. Then a can neither be left or right invertible in $\tilde{A}$, since $\tilde{A} \subseteq \tilde{B}$. So $\tilde{A}a =< a >$ is a proper left ideal in $\tilde{A}$, and $E = \tilde{A}/ < a >$ is an A-module. (Throughout this paper, "module" will be synonymous with "representation".) There can be no B-module H with $H \supseteq E$, since this

would imply $a[1]=[0]$ and $a^{-1}a[1]=[1]$. We say that A is *spectral invariant* in B if $\mathrm{inv}\tilde{A}=\mathrm{inv}(\tilde{B})\cap\tilde{A}$, or in other words if every $a\in\tilde{A}$ is invertible in $\tilde{A}$ if and only if a is invertible in $\tilde{B}$, or equivalently $spec_A(a)=spec_B(a)$. We have just proved that a positive answer to Question A implies spectral invariance.

An example of a non spectral invariant pair $A\subseteq B$ is given by $B=C([-1,1])\supseteq A=\mathcal{A}(D)$, where $\mathcal{A}(D)$ is the algebra of holomorphic functions on the open unit disc in the complex plane, with continuous extension to the boundary. The inclusion map is restriction to the interval $[-1,1]$. The subalgebra A is dense because it contains 1 and the identity function $id(z)=z$ (or use the Stone-Weierstrass theorem). (In fact, A is also a $\star$-subalgebra of B, with $f^\star(z)=\overline{f}(\overline{z})$.) The function $f(z)=z-i$ is invertible in B, but not in A, and the representation of A on $\mathbb{C}$ given by $fv=f(i)v$ does not extend to a representation of B.

To find examples with a positive answer to Question A, we would therefore look at cases when $A\subseteq B$ is spectral invariant. Example 2.1 above is such an example since $f\in C^\infty(M)$ is invertible (in either algebra) if and only if $f(x)\neq 0$ for all $x\in M$.

Consider the case $M=[0,1]$, the unit interval. (To define $C^\infty[0,1]$, we

require one-sided differentiability at the boundary points.) Make $E = \mathbb{C}^2$ an A-module with action

$$\varphi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \varphi(0) & \varphi'(0) \\ 0 & \varphi(0) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{1}$$

Assume for a contradiction that E extends to a continuous B-module. Since A is also dense in $C^1[0,1] \subseteq B$, the action of $C^1[0,1]$ on E must be given precisely by (1), by continuity. Let $\varphi_n(z) = z^{1+1/n}$. Then $\varphi_n \in C^1(S^1) \subseteq B$, $\varphi_n(0) = 0$, $\varphi_n'(z) = (1/n+1)z^{1/n}$, and $\varphi_n'(0) = 0$. Therefore

$$\varphi_n \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \varphi_n(0) & \varphi_n'(0) \\ 0 & \varphi_n(0) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2}$$

But $\varphi_n \to z$ in sup norm (the norm on B), and

$$z \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}. \tag{3}$$

This contradicts the continuity of the extension. In particular, there is no extension to a $\star$-representation of B on a Hilbert space, so the answer to Question A is still "no" in this very simple case, where A *is* spectral invariant in B. (It is interesting to note that the kernel of this representation of A is in fact a closed $\star$-ideal in A.)

The problem in making a continuous extension appears to be that by

allowing E to have nontrivial invariant subspaces (i.e. $\begin{pmatrix} \mathbb{C} \\ 0 \end{pmatrix}$), one gets representations which are not point evaluations.[1] We modify Question A and replace it with:

Question B. When is every irreducible A-module contained in a $\star$-representation of B on a Hilbert space?

If irreducible means "algebraically irreducible", the answer to Question B is "yes" if and only if A is spectral invariant in B [Sch1, 1992].

Definition 2.3. In this paper, by an *irreducible* A-module we mean a topologically irreducible, continuous Fréchet A-module. We say that A is an *m-convex Fréchet algebra* if the algebra A is a Fréchet space such that $\|ab\|_n \leq C_n \|a\|_n \|b\|_n$ for some choice of seminorms $\{\| \quad \|_n\}_{n=0}^{\infty}$ for A, and constants $C_n > 0$ [Mi, 1952]. The inclusion $A \hookrightarrow B$ is always assumed continuous. An A-module E is a *Fréchet A-module* if E is a Fréchet space such that the action $A \times E \to E$ is jointly continuous. *Topologically irreducible*

[1] Another route is to use the $\star$-operation on A, and require E to be a Hilbert space with a $\star$-represention of A. It follows from the spectral invariance of A in B that B must be the enveloping $C^\star$-algebra of A (the strongest $C^\star$-completion of A). In the case that the representation of A is required to be by bounded operators, then *all* such representations would extend to B (on the same Hilbert space), and the problem becomes trivial.

means that E has no closed A-invariant subspaces besides $\{0\}$ and E. The containment $E \hookrightarrow \mathcal{H}$ of E into the $\star$-representation of B on a Hilbert space $\mathcal{H}$ is required to be continuous.

In all the examples in this paper, A will be an m-convex Fréchet algebra. Question B *does* have a positive answer in the case $B = C(M) \supseteq A = C^\infty(M)$ [duC1, 1989]. The argument is briefly as follows. Let E be an irreducible A-module. Arguing as for irreducible representations of $C(M)$, show that there exists $p \in M$ such that the representation factors through the quotient $A/\{f \in A \mid f^{(k)}(p) = 0, \quad k = 0, 1, 2, \ldots\}$. This quotient happens to be isomorphic to the Fréchet algebra of power series in n variables $\mathbb{C}[x_1, \ldots, x_n]$, where n is the dimension of the manifold [Tr, 1967], Theorem 38.1. (The identification is given by the Taylor series expansion of $f \in A$ at the point p, in the indeterminates $x_1, \ldots, x_n$.) But $\mathbb{C}[x_1, \ldots, x_n]$ has the unique maximal ideal $< x_1, \ldots, x_n >$, so by irreducibility the representation factors through the quotient of $\mathbb{C}[x_1, \ldots, x_n]$ by $< x_1, \ldots, x_n >$, which is just $\mathbb{C}$. Hence E is just $\mathbb{C}$ with action $fz = f(p)z$ for some $p \in M$, which clearly extends to a $\star$-representation of $C(M)$ on $\mathbb{C}$.

For Example 2.2, the answer to Question B (as well as Question A) is still "no". Define $E = l^2(\mathbb{N}, \frac{1}{1+n^2}) = \{\xi \colon \mathbb{N} \to \mathbb{C} \mid \|\xi\|_2 = \left(\sum_n |\xi(n)|^2 \frac{1}{1+n^2}\right)^{1/2} <$

$\infty\}$, with module action $(T\xi)(n) = \Sigma_m T(n,m)\xi(m)$, for $T \in \mathcal{K}^\infty$, $\xi \in E$. This is easily seen to be topologically irreducible and continuous. However, E is "too big" to be contained in a Hilbert space representation of $\mathcal{K}$. Assume $\mathcal{H} \supseteq E$ for a contradiction. Using the matrix units in $\mathcal{K}$, note that $n, m \mapsto < \epsilon_n, \epsilon_m >_{\mathcal{H}}$ must be $c\delta_{nm}$ for some constant $c > 0$, where ϵ_n is the step function at $n \in \mathbb{N}$. Note that $1 = \sum \epsilon_m$ converges absolutely in E, and so in $\mathcal{H}$. Hence $< 1, 1 > = < \sum \epsilon_n, \sum \epsilon_m > = \sum c = \infty$, a contradiction. We "tighten up" the allowable E's by replacing Question B with:

Question C. Is every algebraically cyclic subrepresentation F of every irreducible representation E of A contained in a $\star$-representation of B on a Hilbert space? Here *algebraically cyclic subrepresentation* means that there is some $e \in E$ such that $F = Ae \subseteq E$, *and* $e \in F$. We give F the quotient topology from A.

Note that F itself is not required to be irreducible. However, the requirement that F be contained in the irreducible representation E is sufficient to give a "yes" answer to Question C in both Examples 2.1 and 2.2 [duC1, 1989]. We will outline the proof for $\mathcal{K}^\infty$ in the proof of Theorem 5.3 below.

3 A "yes" answer still implies spectral invariance

As we noted above [Sch1, 1992], if A is not spectral invariant in B, then there is an *algebraically* irreducible A-module not contained in any B-module. In this section, we show that this is also true for algebraically cyclic submodules of topologically irreducible A-modules. First consider the case when A is a Banach algebra. Let $a \in \tilde{A}$, $a^{-1} \in \tilde{B} - \tilde{A}$. Then a is not left invertible in $\tilde{A}$, $< a >$ is a proper left ideal in $\tilde{A}$, and by Zorn's lemma $< a >$ is contained in a maximal ideal N. (N is closed since $\tilde{A}$ has an open group of invertibles.) Then $E = \tilde{A}/N$ is a topologically irreducible A-module with no extension to a B-module. In fact, E is algebraically irreducible, and so an algebraically cyclic submodule of itself. When A is an m-convex Fréchet algebra which is not spectral invariant in B, the group of invertibles may not be open, but the result is still true:

Theorem 3.1. *Let A be a dense m-convex Fréchet subalgebra of a Banach algebra B. Assume that for every topologically irreducible Banach A-module E, we know that every algebraically cyclic submodule of E is contained in a B-module. Then A is spectral invariant in B.*

Proof: First redefine the norms $\{\| \quad \|_n\}_{n=0}^{\infty}$ on $\tilde{A}$ so that they are increasing, and arrange that $\| \quad \|_0$ is the norm on B. Let A_n be the completion of $\tilde{A}$ in the nth norm $\| \quad \|_n$, and let $A_0 = \tilde{B}$, $A_\infty = \tilde{A}$. By the sub-multiplicativity of $\| \quad \|_n$, each A_n is a Banach algebra. If $n, m \in \mathbb{N} \cup \{\infty\}$ and $m \geq n$, let $\pi_{nm} \colon A_m \to A_n$ be the canonical map induced from the identity map from $\tilde{A}$ to $\tilde{A}$. Then $\pi_{kn} \circ \pi_{nm} = \pi_{km}$ and $\|\pi_{nm}(a)\|_n \leq \|a\|_m$ for $a \in A_m$.

If $a_n \in A_n$ is a sequence such that $\pi_{nm}(a_m) = a_n$ for each $m, n \in \mathbb{N}$, we show that there is an $a \in A_\infty$ such that $\pi_{n\infty}(a) = a_n$. (This is also done in [Mi, 1952], Theorem 5.1.) Since $\pi_{n\infty}(A_\infty)$ is dense in A_n, we may choose $a_{(n)} \in A_\infty$ such that $\|\pi_{n\infty}(a_{(n)}) - a_n\|_n < 1/n$. Then if $n \geq m$,

$$\begin{aligned} \|a_m - \pi_{m\infty}(a_{(n)})\|_m &= \|\pi_{mn}(a_n) - \pi_{mn} \circ \pi_{n\infty}(a_{(n)})\|_m \\ &\leq \|a_n - \pi_{n\infty}(a_{(n)})\|_n < 1/n. \end{aligned} \tag{4}$$

So for each $m \in \mathbb{N}$, $\pi_{m\infty}(a_{(n)}) \to a_m$ in A_m. It follows that $a_{(n)}$ is Cauchy in each norm, and so Cauchy in A_∞. Hence $a_{(n)} \to a$ in A_∞ for some $a \in A_\infty$. Clearly $\pi_{n\infty}(a) = a_n$, and it follows that $A_\infty \cong \{[a_n] \in \prod_{n=0}^{\infty} A_n \mid \pi_{nm}(a_m) = a_n\}$ is an isomorphism of Fréchet algebras.

Since A is not spectral invariant in B, there is some $a \in A_\infty$ such that $a^{-1} \in A_0 - A_\infty$. We show that a is not invertible in some A_n for some $n \in \mathbb{N}$. (This argument is taken from [Mi, 1952], Theorem 5.2, (c).) Assume a is

invertible in each A_n, and let a_n be the inverse of a in A_n. Then for $m \geq n$, $\pi_{nm}(a_m)a = \pi_{nm}(a_m a) = \pi_{nm}(1) = 1$, and similarly $a\pi_{nm}(a_m) = 1$, so by the uniqueness of two sided inverses in A_∞, $a_n = \pi_{nm}(a_m)$. By the previous paragraph, it follows that there is a $b \in A_\infty$ for which $\pi_{n\infty}(b) = a_n$. Then b is an inverse for a. But $a^{-1} \notin A_\infty$, so we conclude that a is not invertible in some A_n.

We assume that a is not left invertible in A_n. (If a is not right invertible, the same construction works with right modules in place of left modules.) Let N be a maximal left ideal in A_n containing a. Since any unital Banach algebra has an open group of invertible elements, N is closed since it is maximal. We thus have a continuous algebraically irreducible Banach A_n-module $E = A_n/N$. Since A_∞ is dense in A_n with stronger topology, E is an irreducible A_∞-module. Thus E is an irreducible Banach A-module.

If ξ is the coset of the identity in E, then $F = A\xi$ gives an algebraically cyclic subrepresentation of A. We must check that ξ actually lies in F. If A is unital, this is obvious. Otherwise, we have $a = \lambda + a'$ for some $a' \in A$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Then $\xi = [-a'/\lambda] \in F$. To see that F has no extension to a representation of B, note that $a\xi = 0$ but a is invertible in $\tilde{B}$. $\square$

Remarks: If all the maps $\pi_{n0}\colon A_n \to B$ are injective in the proof of Theorem

3.1, or if A is a $\star$-subalgebra of B, then it suffices to consider only left (or right) A-modules. We can also generalize Theorem 3.1 considerably. The proof does not require that A be dense in B. Also, B could be any topological algebra, as long as some finite number of norms on A induces a topology stronger than the topology on B. In fact, $\tilde{B}$ does not have to have an open group of invertible elements.

4 The smooth irrational rotation algebra, and other cases when representations do not extend

Our notation for the $C^\star$-algebra associated to a dynamical system G, M will be $C^\star(G, M)$, and $\mathcal{S}(G, M)$ will denote a smooth subalgebra of Schwartz functions on $G \times M$.

Example 4.1. Let $\mathbb{T} = S^1$ denote the circle, viewed as $[0, 1]$ with endpoints identified. Let $A_\theta^\infty = \mathcal{S}(\mathbb{Z}, \mathbb{T}) = \mathcal{S}(\mathbb{Z}, C^\infty(\mathbb{T}))$, where $\mathbb{Z}$ acts via $\alpha_p(z) = z + p\theta$. This is a unital, dense m-convex Fréchet $\star$-subalgebra of the irrational rotation $C^\star$-algebra $A_\theta = C^\star(\mathbb{Z}, \mathbb{T})$. We know that A_θ^∞ is spectral invariant

in A_θ since it is the set of C^∞-vectors for the following action of $\mathbb{T}^2$ on A_θ:

$$\alpha_{(z_1,z_2)}(F)(n,z) = e^{2\pi i z_1 n} F(n, z+z_2), \qquad z_1, z_2, z \in \mathbb{T}, \quad n \in \mathbb{Z}. \tag{5}$$

We construct a topologically irreducible Banach space representation of A_θ^∞, which has an algebraically cyclic subrepresentation not contained in any $\star$-representation of A_θ on a Hilbert space. Let A_θ^∞ act on $E = L^1(\mathbb{T})$ via

$$(F\psi)(z) = \sum_{n\in\mathbb{Z}} F(n,z)\psi(z - n\theta), \qquad z \in \mathbb{T}. \tag{6}$$

Then E is a Banach A_θ^∞-module. We show that E is topologically irreducible. Both $C^\infty(\mathbb{T}) \subseteq A$ and $L^\infty(\mathbb{T})$ act continuously on E by pointwise multiplication, and by the density of $C^\infty(\mathbb{T})$ in $L^\infty(\mathbb{T})$ in the strong operator topology on E, it suffices to show that $\mathbb{Z}$, $L^\infty(\mathbb{T})$ has no non-trivial closed invariant subspaces, or that every nonzero $\eta \in E$ is cyclic. Let S be the Borel set $\{z \in \mathbb{T} \mid |\eta(z)| > \epsilon\}$, where $\epsilon > 0$ is sufficiently small that S has nonzero measure. Then the characteristic function χ_S is in $L^\infty(\mathbb{T})\eta$. Let $S_N = \cup_{|n|\leq N}\{S + n\theta\}$. Then χ_{S_N} is in the $\mathbb{Z}$, $L^\infty(\mathbb{T})$ span of χ_S. Also $\chi_{S_N} \to \chi_{S_\infty}$ in E. It is well known that the action of $\mathbb{Z}$ on $\mathbb{T}$ by an irrational rotation is ergodic; therefore $\mu(S_\infty)\mu(\mathbb{T} - S_\infty) = 0$ since S_∞ is $\mathbb{Z}$-invariant. But $\mu(S_\infty) \geq \mu(S) > 0$, so $\mu(\mathbb{T} - S_\infty) = 0$. Hence $\chi_{S_\infty} = 1 \in \overline{A\eta}^E$, and so $\overline{A\eta}^E = E$.

Let ξ be any nonnegative element of E which is infinitely differentiable except at some point $p \in \mathbb{T}$, and which grows like $|z-p|^{-1/2}$ in a neighborhood of p. Then $\xi \in L^1(\mathbb{T}) - L^2(\mathbb{T})$. Let F be the algebraically cyclic submodule $A_\theta^\infty \xi$.

We show that $1 \in F$. Let $f \in C^\infty(\mathbb{T})$ be any function for which $f\xi$ is not identically zero, and is always nonnegative and differentiable on $\mathbb{T}$ (such functions exist since ξ is differentiable and nonzero on some interval). By the compactness of $\mathbb{T}$, the sum of finitely many translates (by multiples of θ) of $f\xi$ will be nonvanishing on $\mathbb{T}$. Let ψ be this sum. Then $\psi \in F$ and $1/\psi \in C^\infty(\mathbb{T})$ so $1 \in F$.

Assume for a contradiction that there is a $\star$-representation of A_θ on a Hilbert space $\mathcal{H}$ containing F. Define a $\mathbb{Z}$-invariant positive linear functional on $C(\mathbb{T})$ by $x^\star(f) =< f1, 1 >$, where $< \ , \ >$ is the inner product on $\mathcal{H}$. Then since the translates $f_{z_n} \to f_{z_0}$ in $C(\mathbb{T})$ if $z_n \to z_0$, and $\mathbb{Z}$-orbits are dense in $\mathbb{T}$, we see that $x^\star(f_z) = x^\star(f)$ for $z \in \mathbb{T}$. By the Riesz Representation Theorem, [Ru, 1966], Theorem 2.14,

$$x^\star(f) = \int_{\mathbb{T}} f d\mu \tag{7}$$

for some translation invariant positive Borel measure μ on $\mathbb{T}$. By uniqueness of Haar measure, μ must be Lebesgue measure on the circle (times a

constant).

Let $f_n \in C^\infty(\mathbb{T})$ be functions $0 \leq f_n \leq 1$ satisfying

$$f_n(z) = \begin{cases} 0 & |z-p| \leq 1/(n+1) \\ 1 & |z-p| \geq 1/n. \end{cases} \tag{8}$$

Then since $f_n \in C^\infty(\mathbb{T})$ and ξ grows like $|z-p|^{-1/2}$ near p, we know that

$$< f_n\xi, f_n\xi >=< |f_n\xi|^2 1, 1 >= x^\star(|f_n\xi|^2) = \int_{\mathbb{T}} |f_n\xi|^2 d\mu \longrightarrow \infty \tag{9}$$

as $n \longrightarrow \infty$. (The first step used $f_n\xi \in C(\mathbb{T})$.) Since the representation of A_θ on $\mathcal{H}$ is continuous, and $\{f_n\}$ is a bounded set of elements of A_θ, we must have $< f_n\xi, f_n\xi >$ bounded, which is a contradiction. Thus there can be no $\star$-representation of A_θ on a Hilbert space which extends the representation of A_θ^∞ on F.

Since A_θ^∞ is spectral invariant in A_θ, this shows that the converse of Theorem 3.1 is not true. Note that since spectral invariance is equivalent to the existence of extensions for simple modules, F cannot be algebraically irreducible.

Example 4.2. We give a few examples of what happens with a dense subalgebra of a GCR (or Type I) $C^\star$-algebra which is not CCR. Let M be the

one point compactification of the integers, and let $\mathbb{Z}$ act on M by

$$\alpha_n(z) = \begin{cases} z+n & z \in \mathbb{Z} \\ \\ z & z = \infty. \end{cases} \tag{10}$$

The $C^\star$-crossed product $B = C^\star(\mathbb{Z}, M)$ is not CCR since the orbit $\mathbb{Z}$ is not closed in M [Wi, 1981]. However, it is GCR since $M/\mathbb{Z}$ is T_0 [Go, 1973]. Let $A = \mathcal{S}(\mathbb{Z}, C(M))$.

Let $E = C_0(\mathbb{Z})$. Then E is a closed two-sided $\mathbb{Z}$-invariant ideal in $C(M)$ (with isometric and hence tempered action of $\mathbb{Z}$ by translation), so we may view E as a Banach A-module.

We show that E is a topologically irreducible A-module. Let η be any nonzero element of E. Then by multiplying by some appropriate element of $C(M) \subseteq A$, we may assume that $\eta = \delta_n$ for some $n \in \mathbb{Z}$, where

$$\delta_n(m) = \begin{cases} 1 & m = n \\ \\ 0 & m \neq n \end{cases}, \qquad m \in \mathbb{Z}. \tag{11}$$

Letting $\mathbb{Z}$ act on η shows that every finitely supported element of E is in $A\eta$. Since $C_c(\mathbb{Z})$ is dense in E, this shows that E is topologically irreducible.

Let $\xi \in E$ be such that $\xi \notin l^2(\mathbb{Z})$. (For example, take a weighted sum of step functions, where the weight of the nth step function is $1/|n|^{1/2}$.) Let

$F = A\xi$ be the corresponding algebraically cyclic subrepresentation. (Since $1 \in A$, we have $\xi \in F$.) Note that F contains every element of $C_c(\mathbb{Z})$.

Assume that F is contained in a $\star$-representation π of B on a Hilbert space $\mathcal{H}$. Since the representation of $\mathbb{Z}$ on $\mathcal{H}$ is unitary, we have

$$< \delta_n, \delta_m >=< \pi(k)\delta_n, \pi(k)\delta_m >=< \delta_{n+k}, \delta_{m+k} >, \qquad n, m, k \in \mathbb{N}, \quad (12)$$

where $< \quad , \quad >$ is the inner product on $\mathcal{H}$. Also, since the representation of $C(M)$ on $\mathcal{H}$ is a $\star$-representation, we have

$$< \delta_n, \delta_m >=< \pi(\delta_n)\delta_n, \delta_m >=< \delta_n, \pi(\delta_n)\delta_m >= 0, \qquad m \neq n. \qquad (13)$$

It follows that

$$< \delta_n, \delta_m >= \begin{cases} c & n = m \\ 0 & n \neq m \end{cases} \qquad (14)$$

for some $c > 0$, so the inner product on $\mathcal{H}$, on elements of $C_c(\mathbb{Z})$, is precisely the inner product on $l^2(\mathbb{Z})$.

Let $\varphi_n \in C(M)$ be equal to 1 in the interval $[-n, n]$, and equal to zero outside of $[-n, n]$. Then $\delta_0 \otimes \varphi_n$ is a bounded sequence in B, so $< \varphi_n\xi, \varphi_n\xi >\leq D < \xi, \xi >$, for some constant D. Since $\varphi_n\xi \in C_c(\mathbb{Z})$, we know $< \varphi_n\xi, \varphi_n\xi >= \sum_{k=-n}^{n} c|\xi(k)|^2$. This tends to ∞ since $\xi \notin l^2(\mathbb{Z})$, so we have a contradiction. Hence there is no $\star$-representation of B on a

Hilbert space $\mathcal{H}$ containing F.

For another (similar) GCR, non-CCR example, consider the unitization of $\mathcal{K}^\infty$. Then the algebraically cyclic submodule $(\tilde{\mathcal{K}}^\infty) \cdot 1$ of the E defined near the end of §2 is contained in no $\star$-representation of $\tilde{\mathcal{K}}$ (or $\mathcal{K}$) on a Hilbert space, by the argument in §2. As in Example 4.1, both the dense subalgebras A in Example 4.2 and $\tilde{\mathcal{K}}^\infty$ are spectral invariant, again showing that the converse of Theorem 3.1 is not true.

5 Differentiable representations

Definition 5.1. We say that a Fréchet A-module E is *non-degenerate* (*differentiable*) if $\{v \in E \mid Av = 0\} = \{0\}$ and the image of the canonical map $A \widehat{\otimes} E \to E; a \otimes e \mapsto ae$ is dense (onto) [duC2, 1991]. (All tensor products will be completed in the projective topology.) We make the same definition for right modules, and say that A is *self-differentiable* if A is differentiable both as a left and right module over itself.

Note that if A is unital, every A-module is differentiable. In the case $A = C_c^\infty(G)$, the convolution algebra of compactly supported C^∞-functions on a Lie group G, an A-module E is differentiable if and only if the underlying

action of G on E is C^∞ [DM, 1978].

One advantage to using differentiable representations is that that Morita equivalences work out. For example, let D be a subgroup of $\mathbb{Z}^2$. Then the smooth crossed product $A_1 = \mathcal{S}(\mathbb{Z}^2, \mathbb{Z}^2/D)$ (with action by translation) and the convolution algebra $A_2 = \mathcal{S}(D)$ have the differentiable A_1-A_2 bimodule $X = \mathcal{S}(\mathbb{Z}^2)$. From this one obtains a natural (Morita) equivalence of the category of differentiable A_1-modules to the category of differentiable A_2-modules, which preserves topological irreducibility. For the corresponding $C^\star$-algebras $B_1 = C^\star(\mathbb{Z}^2, \mathbb{Z}^2/D)$ and $B_2 = C^\star(D)$ it is well known that $Y = C^\star(\mathbb{Z}^2)$ is an B_1-B_2 equivalence bimodule [Ri, 1974], giving an equivalence of the category of $\star$-representations of B_1 with the category of $\star$-representations of B_2, which preserves irreducibility. It is not hard to check that the two equivalences preserve extensions. If $E \to \mathcal{H}$ is a morphism of a differentiable A_2-module E into a $\star$-representation of B_2 on $\mathcal{H}$, then $X \widehat{\otimes}_{A_2} E \to Y \overline{\otimes}_{B_2} \mathcal{H}$ is a morphism of a differentiable A_1-module to a $\star$-representation of B_1. Note that if E is topologically irreducible, then so is $X \widehat{\otimes}_{A_2} E$, and both morphisms into the Hilbert spaces must be injective.

If E is a non-degenerate A-module, we let $E_s(A)$ be the image of the canonical map $A \widehat{\otimes} E \to E$ [duC2, 1991]. Then $E_s(A)$ inherits the quotient

topology from $A\widehat{\otimes}E$, making $E_s(A)$ a Fréchet A-module. When A is self-differentiable, the A-module $E_s(A)$ is always differentiable. (Use the fact that the canonical maps $A\widehat{\otimes}E \to E$ and $A\widehat{\otimes}A \to A$ are both onto, so that $A\widehat{\otimes}(A\widehat{\otimes}E) \to E$ is onto, and then factor through to the quotient.)

Lemma 5.2. *Let A be a self-differentiable m-convex Fréchet algebra. Then every algebraically cyclic submodule of an irreducible A-module is contained in a differentiable irreducible A-module.*

Proof: Let E be an irreducible A-module. Let F be any nonzero A-invariant closed subspace of $E_s(A)$. Since the closure of F in E is E by irreducibility, the canonical map $A \otimes F \to E_s(A)$ must have dense image. But the image is contained in F, so $F = E_s(A)$. Thus $E_s(A)$ is irreducible. Every algebraically cyclic submodule of E is the image of a set of the form $A \otimes \{\xi\}$ via the canonical map, and hence contained in $E_s(A)$. $\square$

Thus a positive answer to the following question will imply a positive answer to Question C (and therefore also imply spectral invariance):

Question D. Is every differentiable irreducible representation E of A contained in a $\star$-representation of B on a Hilbert space $\mathcal{H}$?

Theorem 5.3. *Let $\vec{v}_1, \vec{v}_2$ be two vectors in $\mathbb{R}^2$. Let $\mathbb{Z}^2$ act on $\mathbb{R}^2$ via*

$$\alpha_{\vec{n}}(\vec{r}) = \vec{r} + n_1\vec{v}_1 + n_2\vec{v}_2. \tag{15}$$

Let $B = C^\star(\mathbb{Z}^2, \mathbb{R}^2)$ be the $C^\star$-algebra associated with the dynamical system, and let $A = \mathcal{S}(\mathbb{Z}^2, \mathbb{R}^2)$ be the canonical dense (self-differentiable and m-convex) Fréchet subalgebra of Schwartz functions. The following are equivalent.

(i) Every differentiable topologically irreducible A-module is contained in a $\star$-representation of B on a Hilbert space.

(ii) Every algebraically cyclic subrepresentation of every topologically irreducible A-module is contained in a $\star$-representation of B on a Hilbert space.

(iii) $\vec{v}_1$ is not an irrational multiple of $\vec{v}_2$.

(iv) All the $\mathbb{Z}^2$-orbits are closed.

(v) B is CCR.

Remark: Theorem 5.3 does not include any cases when B is GCR but not CCR, but Example 4.2 above ($\mathbb{Z}$ acting on its one-point compactification) shows that representations may not extend in such cases.

Proof: *(iii)⇒(i)* First assume that $\vec{v}_1$ and $\vec{v}_2$ do not lie on the same line. Then every orbit is a translate of a (possibly slanted) copy of $\mathbb{Z}^2$, and a discrete subgroup of $\mathbb{R}^2$. Let $\Omega \subset \mathbb{R}^2$ be open and $\mathbb{Z}^2$-invariant. Define

$$J_\Omega = \{f \in A \mid \mathrm{supp}(F(\vec{n}, \cdot)) \subseteq \Omega, \vec{n} \in \mathbb{Z}^2\}. \tag{16}$$

By the formula for convolution multiplication:

$$F * G(\vec{n}, \vec{r}) = \sum_{\vec{m} \in \mathbb{Z}^2} F(\vec{m}, \vec{r}) G(\vec{n} - \vec{m}, \vec{r} + m_1 \vec{v}_1 + m_2 \vec{v}_2), \tag{17}$$

we see that J_Ω is always a two-sided ideal in A (though rarely closed). Also by this formula, note that if $\Omega_1 \cap \Omega_2 = \phi$, then $J_{\Omega_1} J_{\Omega_2} = 0$.

Consider the parallelogram P in $\mathbb{R}^2$ spanned by $\vec{v}_1$ and $\vec{v}_2$. Then distinct points in the interior P^0 give rise to disjoint $\mathbb{Z}^2$-orbits, and any two distinct points p_1, p_2 can be separated by disjoint open subsets U_1, U_2 of P^0. The $\mathbb{Z}^2$-orbits of U_1, U_2 give rise to disjoint $\mathbb{Z}^2$-invariant open sets Ω_1, Ω_2. Let E be a differentiable irreducible Fréchet A-module. Note that $J_{\Omega_i} E$ is either dense in E, or 0, by topological irreducibility. Thus if $J_{\Omega_1} J_{\Omega_2} E = 0$, then $J_{\Omega_i} E = 0$ for some i. A similar argument works if one or more of the points lies on the boundary of P.

Using a partition of unity, we can show that if $J_{\Omega_\alpha} E = 0$ for $\{\Omega_\alpha\}$ a family of open $\mathbb{Z}^2$-invariant sets, then $J_{\cup \Omega_\alpha} E = 0$. Thus there is some largest open $\mathbb{Z}^2$-invariant open set Ω_{max} such that $J_{\Omega_{max}} E = 0$. By the preceding paragraph (and by maximality), the complement of Ω_{max} cannot contain any $\mathbb{Z}^2$-invariant set with more that one orbit - in other words, $\Omega_{max} = orb^c$ for some orbit orb. Let $J = J_{\Omega_{max}}$.

An easy argument taking limits shows that

$$\overline{J}^A = \{F \in A \mid \partial^l_{r_1}\partial^k_{r_2}F(\vec{n}, |_{orb}) = 0, k, l = 0, 1, \dots, \quad \vec{n} \in \mathbb{Z}^2\}. \tag{18}$$

Also by taking limits, E factors to an $A/\overline{J}$-module. Let

$$I = \{f \in \mathcal{S}(\mathbb{R}^2) \mid \partial^l_{r_1}\partial^k_{r_2}f(|_{orb}) = 0, k, l = 0, 1, \dots\}. \tag{19}$$

Then $\mathcal{S}(\mathbb{R}^2)/I \cong \mathcal{S}(orb)\widehat{\otimes}\mathbb{C}[[x, y]]$. using an argument similar to the one in §2 for $C^\infty(M)$. (An isomorphism $\mathcal{S}(\mathbb{R}^2)/I \to \mathcal{S}(orb)\widehat{\otimes}\mathbb{C}[x, y]$ is given by $[f] \mapsto \sum_{k,l\geq 0}(\partial^k_{r_1}\partial^l_{r_2}f)(\vec{r})x^ky^l$, $\vec{r} \in orb$.) Then

$$A/\overline{J} \cong \mathcal{S}(\mathbb{Z}^2, A/I) \cong \mathcal{S}(\mathbb{Z}^2, orb)\widehat{\otimes}\mathbb{C}[[x, y]] \tag{20}$$

as Fréchet spaces. Note that

$$\begin{aligned} \alpha_{\vec{n}}(\partial^l_{r_1}\partial^k_{r_2}\varphi)(\vec{r} \in orb) &= (\partial^k_{r_1}\partial^l_{r_2}\varphi)(\vec{r} - n_1\vec{v}_1 - n_2\vec{v}_2) \\ &= (\partial^k_{r_1}\partial^l_{r_2}\alpha_{\vec{n}}(\varphi))(\vec{r}), \end{aligned} \tag{21}$$

by linearity, so the Fréchet algebra structure on the tensor product in (20) is just the natural tensor product of Fréchet algebras. Thus E factors to an irreducible $\mathcal{S}(\mathbb{Z}^2, orb)$-module. But $\mathcal{S}(\mathbb{Z}^2, orb) \cong \mathcal{S}(\mathbb{Z}^2, \mathbb{Z}^2)$, with $\mathbb{Z}^2$ acting by translation, which is isomorphic to $\mathcal{K}^\infty$ via $(\theta F)(\vec{n}, \vec{m}) = F(-\vec{m}, \vec{n} - \vec{m})$.

We recall the classification of differentiable $\mathcal{K}^\infty$-modules from [duC1, 1989]. Since $\mathcal{K}^\infty\widehat{\otimes}E \to E$ is onto, E is a quotient of the $\mathcal{K}^\infty$-module

$\mathcal{K}^\infty \widehat{\otimes} E = \mathcal{S}(\mathbb{Z}, F)$, where $F = \mathcal{S}(\mathbb{Z}, E)$. $\mathcal{K}^\infty$ acts only on the $\mathcal{S}(\mathbb{Z})$ part of $\mathcal{S}(\mathbb{Z}, F)$. Let E_1 be a closed $\mathcal{K}^\infty$-submodule of $\mathcal{S}(\mathbb{Z}, F)$. Let $\epsilon_0 \in \mathcal{S}(\mathbb{Z})$ be the step function at $0 \in \mathbb{Z}$. Let $F_1 = \{f \in F \mid \epsilon_0 \otimes f \in E_1\}$. Identify F_1 with $\epsilon_0 \otimes F_1 \subseteq \mathcal{S}(\mathbb{N}, F)$. Then $e_{n0}F_1 \subseteq E_1$ for every n, so $\mathcal{S}(\mathbb{Z}) \otimes F_1 \subseteq E_1$. Closing in the Schwartz topology in $\mathcal{S}(\mathbb{Z}, F)$, we get $\mathcal{S}(\mathbb{Z}, F_1) = E_1$. We have proved that E is of the form $\mathcal{S}(\mathbb{Z}, F)/\mathcal{S}(\mathbb{Z}, F_1) = \mathcal{S}(\mathbb{Z}, F/F_1)$. Using irreducibility, $F/F_1 = \mathbb{C}$, and $E = \mathcal{S}(\mathbb{Z})$, with the standard action of $\mathcal{K}^\infty$. This is clearly contained in the standard $\star$-representation of $\mathcal{K}$ on $l^2(\mathbb{Z})$, and completes the proof of *(iii)*$\Rightarrow$ *(i)* when $\vec{v}_1$ and $\vec{v}_2$ span $\mathbb{R}^2$.

Next, assume that v_1 and v_2 point in the same direction, but that $v_1 = (p/q)v_2$ for some rational number p/q. This is similar to the above case. A differentiable irreducible representation E of A factors through to a differentiable irreducible representation of $\mathcal{S}(\mathbb{Z}^2, \mathbb{R})$, where the first copy of $\mathbb{Z}$ translates by 1, and the second by p/q. Factoring further, we get a representation of $\mathcal{S}(\mathbb{Z}^2, \mathbb{Z}^2/D)$, where D is the isotropy subgroup of some $\mathbb{Z}^2$-orbit on $\mathbb{R}$. By the Morita equivalence results mentioned above, we are reduced to showing that all differentiable irreducible representations of $\mathcal{S}(D) \cong \mathcal{S}(\mathbb{Z})_{conv} \cong C^\infty(\mathbb{T})_{ptwise}$ extend to $\star$-representations of $C^\star(\mathbb{Z})_{conv} \cong C(\mathbb{T})_{ptwise}$ on Hilbert

spaces. But we have seen this in §2. (They are all point evaluations.)

(ii) $\Rightarrow$ *(iii)* Assume that $v_1 = \gamma v_2$, γ irrational. Without loss of generality, we may replace A with the quotient algebra $\mathcal{S}(\mathbb{Z}^2, \mathbb{R})$, where the first copy of $\mathbb{Z}$ translates by 1, and the second by γ. Let $E = L^1(\mathbb{R}, d\mu)$, with μ Lebesgue measure. Let $\mathbb{Z}^2$ act on E by $(\vec{n}\xi)(r) = \xi(r - n_1 - n_2\gamma)$ and let $\mathcal{S}(\mathbb{R})$ act on E by pointwise multiplication. This gives a covariant (tempered) representation, which integrates to a representation of A on E. Standard arguments show that E is a topologically irreducible A-module.

Let $F = A\xi$, for some $\xi \in L^1(\mathbb{R}) - L^2(\mathbb{R})$, ξ supported in the interval $[-1, 1]$, and ξ infinitely differentiable except at 0. Note that $\xi \in F$ since the pointwise multiplication operators $C_c^\infty(\mathbb{R})$ are contained in $\mathcal{S}(\mathbb{R}) \subseteq A$. Also $C_c^\infty(\mathbb{R}) \subseteq F \subseteq E$ since ξ is smooth on an interval.

Assume for a contradiction that F is contained in a $\star$-representation of $C^\star(\mathbb{Z}^2, \mathbb{R})$ on a Hilbert space $\mathcal{H}$. (By replacing $\mathcal{H}$ with the closure of F in $\mathcal{H}$, we may assume that F is dense in $\mathcal{H}$.) Let $\eta \in C_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$ and let $\psi \in C_c^\infty(\mathbb{R}) \subseteq F$ satisfy $\psi(x) = 1$ for x in some neighborhood of the support of η, and $0 \leq \psi \leq 1$. Then we define

$$\Phi(\eta) =< \eta\psi, \psi > . \tag{22}$$

If ψ' is another element of $C_c^\infty(\mathbb{R})$ satisfying the same properties as ψ, it is easily checked that $< \eta\psi, \psi > = < \eta\psi', \psi' >$. (Multiply η by the square of some nonnegative function $\psi'' \in C_c^\infty(\mathbb{R})$ which is equal to one on the support of η and vanishes wherever ψ and ψ' disagree.) So $\Phi\colon C_c^\infty(\mathbb{R}) \to \mathbb{C}$ is a well-defined function. Picking ψ so that $\psi = 1$ in a neighborhood of the support of η_1 and η_2, we see that $\Phi(\eta_1 + \eta_2) = \Phi(\eta_1) + \Phi(\eta_2)$. Similarly $\Phi(c\eta) = c\Phi(\eta)$ for $c \in \mathbb{C}$ and $\Phi(\alpha_{\vec{n}}(\eta)) = \Phi(\eta)$. Moreover, $\Phi(\eta_r) = \Phi(\eta)$ for $r \in \mathbb{R}$, since $\mathbb{Z} + \mathbb{Z}\gamma$ is a dense subgroup of $\mathbb{R}$, and the translates $\eta_{r_n} \to \eta_{r_0}$ in $C_c^\infty(\mathbb{R})$ if $r_n \to r_0$ in $\mathbb{R}$. If $\eta \geq 0$, then η has a square root in $C_0(M)$, so $\Phi(\eta) = < \eta^{1/2}\psi, \eta^{1/2}\psi > \geq 0$. Now use the continuity of the representation of $C_0(\mathbb{R})$ on $\mathcal{H}$ to extend Φ to be defined on $C_c(\mathbb{R})$ and not just $C_c^\infty(\mathbb{R})$. Since Φ is a translation invariant positive linear functional on $C_c(\mathbb{R})$, by the Riesz Representation Theorem [Ru, 1966], Theorem 2.14, there is some $\mathbb{R}$-invariant positive Borel measure ν on $\mathbb{R}$ such that $\Phi(\eta) = \int_{\mathbb{R}} \eta(x) d\nu(x)$. By translation invariance, ν must be (a scalar multiple of) Lebesgue measure. Thus on functions $\eta_1, \eta_2 \in C_c^\infty(\mathbb{R}) \subseteq F$, the inner product on $\mathcal{H}$ is just given by the $L^2(\mathbb{R}, d\mu)$ inner product of η_1 and η_2. Since $C_c^\infty(\mathbb{R})$ is dense in F and hence dense in $\mathcal{H}$, we must have $\mathcal{H} \cong L^2(\mathbb{R})$.

Finally, let $\chi_n \in C_c^\infty(\mathbb{R})$ satisfy $\chi_n(r) = 1$ for $r \in [-1, -\epsilon] \cup [\epsilon, 1]$, $\chi_n(r) =$

0 for r in some neighborhood of zero, $0 \leq \chi_n \leq 1$, and $\mathrm{supp}(\chi_n) \subseteq [-2, 2]$. Then

$$< \chi_n \xi, \chi_n \xi >= \int |\chi_n \xi(r)|^2 dr \to \infty \tag{23}$$

as $n \to \infty$. But χ_n is a bounded sequence in $C_0(\mathbb{R})$, so this gives a contradiction.

So we have *(ii)* $\Rightarrow$ *(iii)* $\Rightarrow$ *(i)*. *(i)* $\Rightarrow$ *(ii)* follows from Lemma 5.2, and *(iii)* $\Leftrightarrow$ *(iv)* is obvious. *(iv)* $\Leftrightarrow$ *(v)* is well-known [Wi, 1981]. $\square$

6 When algebraically cyclic subrepresentations extend, but differentiable representations do not

We construct a dense self-differentiable subalgebra of the compact operators, which is partly smooth and partly not-smooth, and which has a differentiable irreducible representation that does not extend, though each algebraically cyclic subrepresentation of each irreducible representation does extend. This is a case when the answer is "yes" to Question C but "no" to Question D.

§6.1. Let A be the smooth crossed product $\mathbb{Z} \rtimes_\alpha c_0(\mathbb{Z})$, where α is translation. Let A be the Fréchet algebra whose underlying Fréchet space is $\mathcal{S}(\mathbb{Z})\widehat{\otimes}c_0(\mathbb{Z})$, with multiplication given by

$$F * G(n,m) = \sum_{k\in\mathbb{Z}} F(k,m)G(n-k,m-k). \tag{24}$$

Here $\mathcal{S}(\mathbb{Z}) = \{\varphi\colon \mathbb{Z} \to \mathbb{C} \mid \|\varphi\|_p = \sum_k |\varphi(k)|\sigma(k)^p < \infty, \quad p \in \mathbb{N}\}$ with scale

$$\sigma(k) = 1 + |k|, \tag{25}$$

denotes the set of Schwartz functions on $\mathbb{Z}$, and $c_0(\mathbb{Z})$ is the Banach space of sequences on $\mathbb{Z}$ which vanish at ∞. We can make the identification of Fréchet spaces $\mathcal{S}(\mathbb{Z})\widehat{\otimes}c_0(\mathbb{Z}) \cong \mathcal{S}(\mathbb{Z}, c_0(\mathbb{Z}))$ by [Tr, 1967], Theorems 44.1 and 50.1(f). Let B be the $C^\star$-crossed product $\mathbb{Z} \rtimes c_0(\mathbb{Z})$. In the standard representation on the Hilbert space $\ell^2(\mathbb{Z})$,

$$F * \xi(m) = \sum_{k\in\mathbb{Z}} F(k,m)\xi(m-k), \tag{26}$$

B is the $C^\star$-algebra of compact operators.

§6.2. A is a self-differentiable Fréchet algebra. Since A is a smooth crossed product and the scale σ in (25) is submultiplicative, A is an m-convex Fréchet algebra by [Sch2, 1993], Theorem 3.1.7. Also A is self-differentiable

by [Sch4, 1994], Theorem 5.12, since the pointwise-mulitiplication algebra $c_0(\mathbb{Z})$ is self-differentiable. (In fact the algebraic tensor product $c_0(\mathbb{Z}) \otimes c_0(\mathbb{Z})$ maps onto $c_0(\mathbb{Z})$ via the canonical mapping.)

§6.3. An irreducible differentiable representation of A that doesn't extend to B. Let E be the Banach space $c_0(\mathbb{Z})$, and define A's action on E by the same formula as (26). Then E is a differentiable A-module by [Sch4, 1994], Theorem 5.3. It is not hard to show $\delta_0 \in A\xi$ for any nonzero $\xi \in E$, and that finite support functions $c_f(\mathbb{Z})$ are contained in $A\delta_0$, so E is a topologically irreducible A-module. But E cannot extend to a B-module, by an argument similar to the one used in Example 4.2 above.

§6.4. Every irreducible representation of A can be viewed as A acting on functions from $\mathbb{Z}$ to $\mathbb{C}$ through the standard action. Moreover, the functions have polynomial growth. Let E be a topologically irreducible representation of A. For $i, j \in \mathbb{Z}$, note that $e_{ij} = \delta_{i-j} \otimes \delta_i$ are matrix units in A, with $e_{ij} * e_{kl} = \delta_{jk} e_{il}$, $e_{ij}^{\star} = e_{ji}$. Then $p_i = e_{ii} = \delta_0 \otimes \delta_i$ is the ith minimal projection. Since $\mathrm{span}\{p_i\}$ is dense in $\mathbb{C}\delta_0 \otimes c_0(\mathbb{Z})$, and $\mathrm{span}\{e_{0j}\}$ is dense in $\mathcal{S}(\mathbb{Z}) \otimes \mathbb{C}\delta_0$, $\mathrm{span}\{e_{ij}\}$ is dense in A. Let E_i be the

linear subspace p_iE of E. Then $e_{ij}E_k = \delta_{jk}E_i$, and one E_i is zero if and only if all the E_i's are zero. Since $\bigoplus_{i\in\mathbb{Z}} E_i = \big(\mathrm{span}\{e_{ij}\}\big)E$ is dense in E, and $E \neq \{0\}$, we have $E_i \neq \{0\}$.

Let e, f be nonzero elements of E_i. Then $\mathrm{span}\{e_{ij}\}e$ is dense in E, so f is a limit point. Let $a_n \in \mathrm{span}\{e_{ij}\}$ be a sequence for which a_ne converges to f. Then p_ia_ne converges to $p_if = f$. Also $p_ia_np_ie = p_ia_ne$, so $p_ia_np_ie$ converges to f. But p_i is a minimal projection, so $p_ia_np_i$ is a complex number z_n times p_i. So f is a multiple of e and E_i is one-dimensional.

Let e_0 be some nonzero element of E_0, and set $e_i = e_{i0}e_0 \in E_i$ for each $i \in \mathbb{Z}$. Then $e_{ij}e_k = \delta_{jk}e_i$. Define a map $\Phi\colon E \to \big\{\xi \,|\, \xi\colon \mathbb{Z} \to \mathbb{C}\big\}$ by letting $\Phi(e)(i)$ be the complex number z_i for which $p_ie = z_ie_i$. The map Φ is one-to-one since $\Phi(e) \equiv 0$ implies $Ae = 0$ and hence $e = 0$. Through A's action on E,

$$\begin{aligned}
\big(\delta_k \otimes \delta_l\big) * e_i &= \big(\delta_k \otimes \delta_l\big) * e_{i0} * e_0 \\
&= \big(e_{l,l-k} * e_{i0}\big) * e_0 \\
&= \big(\delta_{i,l-k}\big)e_{l,0} * e_0 \\
&= \big(\delta_{i,l-k}\big)e_l,
\end{aligned} \tag{27}$$

where $\delta_k \otimes \delta_l$ is an elementary tensor product in $A = \mathcal{S}(\mathbb{Z})\widehat{\otimes}c_0(\mathbb{Z})$.

Thus we get an action of A on $\Phi(E)$ via

$$\begin{aligned}
\Phi\Big(\big(\delta_k \otimes \delta_l\big) \circ \Phi^{-1} * \xi\Big)(i) &= \Phi\Big(\big(\delta_k \otimes \delta_l\big) * \sum_{j\in\mathbb{Z}} \xi(j)e_j\Big)(i) \\
&= \sum_{j\in\mathbb{Z}} \xi(j)\Phi\Big(\big(\delta_k \otimes \delta_l\big) * e_j\Big)(i) \\
&= \sum_{j\in\mathbb{Z}} \xi(j)\Phi\Big(\delta_{j,l-k}e_l\Big)(i) \\
&= \xi(l-k)\Phi\Big(e_l\Big)(i) \\
&= \delta_{il}\xi(l-k), \qquad (28)
\end{aligned}$$

for $\xi\colon \mathbb{Z} \to \mathbb{C}$ in $\Phi(E)$. For $F \in A$ and $\xi \in \Phi(E)$, this shows $F * \xi$ is the standard action (26).

We show that any $\xi \in \Phi(E)$ can be viewed as a continuous linear functional on $\mathcal{S}(\mathbb{Z})$. Through the action of the smooth crossed product $A = \mathbb{Z} \rtimes c_0(\mathbb{Z})$ on E, we have a natural action of the convolution algebra $\mathcal{S}(\mathbb{Z})$ on E. For $\varphi \in \mathcal{S}(\mathbb{Z})$ and $\xi \in \Phi(E)$, $\varphi * \xi(m) = \sum_{k\in\mathbb{Z}} \varphi(k)\xi(m-k)$. For fixed $\xi \in \Phi(E)$, the map $\varphi \mapsto \sum_{k\in\mathbb{Z}} \varphi(k)\xi(-k) \in \mathbb{C}$ is a continuous linear functional on $\mathcal{S}(\mathbb{Z})$. Thus for some $C > 0$ and $p \in \mathbb{N}$, $| < \varphi, \xi > | \leq C\|\varphi\|_p$, from which it follows that

$$|\xi(k)| \leq C\sigma(k)^p, \qquad k \in \mathbb{N}. \qquad (29)$$

§6.5. Algebraically cyclic subrepresentations of irreducible representation of A. Let E be a topologically irreducible representation of A. Let f be a nonzero element of E such that $F_f * f = f$ for some $F_f \in A$.

Lemma 6.6. *$f \in \mathcal{S}(\mathbb{Z})$.*

Proof: Using the framework of §6.4 and (29), let $C_f > 0$ and $p_f \in \mathbb{N}$ be such that $|f(k)| \leq C_f \sigma(k)^{p_f}$, $k \in \mathbb{Z}$. Let $d \in \mathbb{N}$ be greater than or equal to p_f. Since F_f is in the smooth crossed product A, $\|F_f(\cdot, m)\|_d^\infty \to 0$ as $|m| \to \infty$, where

$$\|F_f(\cdot, m)\|_d^\infty = \sup_{k \in \mathbb{Z}} \Big(\sigma(k)^d |F_f(k, m)| \Big) \qquad \text{for } m \in \mathbb{Z}. \tag{30}$$

Thus

$$\begin{aligned}
|f(m)| &= |F_f * f(m)| \\
&= \Big| \sum_{k \in \mathbb{Z}} F_f(k, m) f(m-k) \Big| \\
&\leq \sum_{k \in \mathbb{Z}} \big| F_f(k, m) \big| \big| f(m-k) \big| \\
&= \sum_{k \in \mathbb{Z}} \frac{1}{\sigma(k)^2} \big| \sigma(k)^{d+2} F_f(k, m) \big| \left| \frac{f(m-k)}{\sigma(k)^d} \right| \\
&\leq \sum_{k \in \mathbb{Z}} \frac{1}{\sigma(k)^2} \|F_f(\cdot, m)\|_{d+2}^\infty \Big| \frac{f(m-k)}{\sigma(k)^d} \Big| \\
&\leq C_{m,d} * \sup_{k \in \mathbb{Z}} \left(\frac{|f(m-k)|}{\sigma(k)^d} \right),
\end{aligned} \tag{31}$$

where $C_{m,d} \to 0$ as $|m| \to \infty$.

Define

$$N_d = \max\{\, |m| \mid C_{m,d} \geq \frac{1}{2}\,\}. \tag{32}$$

Then if $|m| > N_d$, by (31) we have

$$|f(m)| < \frac{1}{2}\frac{|f(m-k_1)|}{\sigma(k_1)^d} \qquad \text{for some } k_1 \in \mathbb{Z}. \tag{33}$$

If $|m - k_1| > N_d$, we can repeat the process

$$\begin{aligned} |f(m)| &< \frac{1}{2}\frac{|f(m-k_1)|}{\sigma(k_1)^d} \\ &< \frac{1}{2}\frac{1}{\sigma(k_1)^d} \quad * \quad \frac{1}{2}\frac{|f(m-k_1-k_2)|}{\sigma(k_2)^d} \\ &= \frac{1}{2^2}\frac{|f(m-k_1-k_2)|}{\big(\sigma(k_1)\sigma(k_2)\big)^d} \qquad \text{for some } k_2 \in \mathbb{Z}. \end{aligned} \tag{34}$$

After l times, we get

$$|f(m)| < \frac{1}{2^l}\frac{|f(m-k_1-\cdots-k_l)|}{\big(\sigma(k_1)\cdots\sigma(k_l)\big)^d}, \tag{35}$$

where $m, m-k_1, \ldots m-k_1-\cdots-k_{l-1}$ all have absolute value greater than N_d.

Since the right hand side of (35) is bounded by

$$\frac{C_f\sigma(m-k_1-\cdots-k_l)^{p_f}}{2^l(\sigma(k_1)\cdots\sigma(k_l))^d} \leq \frac{C_f\sigma(m)^{p_f}}{2^l}, \tag{36}$$

which tends to zero as $l \to \infty$, either $f(m) = 0$ or we can find some $l \in \mathbb{N}$

for which $|m - k_1 - \cdots k_l| \leq N_d$. In the latter case,

$$\begin{aligned}
\sigma(m)^d |f(m)| &< \frac{1}{2^l} \frac{\sigma(m)^d C_f \sigma(N_d)^{p_f}}{\big(\sigma(k_1)\cdots\sigma(k_l)\big)^d} && \text{by (35)} \\
&\leq \frac{1}{2^l} C_f \sigma(N_d)^{p_f+d} \\
&= \frac{1}{2^l} M_{f,d}, && (37)
\end{aligned}$$

where $M_{f,d}$ is a constant independent of l and m, and we used $\sigma(m) \leq \sigma(m - k_1 - \cdots k_l)\sigma(k_1)\cdots\sigma(k_l) \leq \sigma(N_d)\sigma(k_1)\cdots\sigma(k_l)$ in the second step. Note that if $|m| \leq N_d$ to begin with, $\sigma(m)^d|f(m)| \leq C_f\sigma(N_d)^{p_f+d}$ and (37) still holds, with $l = 0$ and $\leq$. So $\|f\|_d^\infty \leq M_{f,d} < \infty$ for any $d \geq p_f$. This shows $f \in \mathcal{S}(\mathbb{Z})$. $\square$

Corollary 6.7. *Any algebraically cyclic subrepresentation of a topologically irreducible representation of A extends to the standard representation of B on $\ell^2(\mathbb{Z})$.*

Proof: Let E and f be as above. By §6.4, f is a function from $\mathbb{Z}$ to $\mathbb{C}$, and A's action on f is the standard action (26). By Lemma 6.6, $f \in \mathcal{S}(\mathbb{Z})$. The kernel of the map $F \in A \mapsto F * f \in E$ is the closed left ideal of A

$$\begin{aligned}
N_f &= \{\, F \in A \,|\, F * f = 0 \,\} \\
&= \left\{ F \in A \,\middle|\, \sum_{k\in\mathbb{Z}} F(k,m) f(m-k) = 0, \quad m \in \mathbb{Z} \right\}, && (38)
\end{aligned}$$

which is the same kernel as when A acts through (26) on $\mathcal{S}(\mathbb{Z})$ on the function $f \in \mathcal{S}(\mathbb{Z})$. The A-module $A{*}f$ is identified with A/N (the topology is induced from A), which is contained in the A-module $\mathcal{S}(\mathbb{Z})$ with action (26). Since $\mathcal{S}(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$ with continuous inclusion, and B's action on $\ell^2(\mathbb{Z})$ is given by (26) as well, the A-module $\mathcal{S}(\mathbb{Z})$ extends to the B-module $\ell^2(\mathbb{Z})$. $\square$

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